For a moment, we broaden our scope to functions of two variables,

$$z = f(x,y)$$

Examples: 1) $f(x,y) = x^2 y$

2) $g(x,y) = \cos(x+y^3)$

The partial derivatives

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} \quad \text{of } f$$

are obtained by, respectively,

differentiating f with respect to x while holding y constant $\left(\frac{\partial f}{\partial x}\right)$ or

differentiating f with respect to y while holding x constant $\left(\frac{\partial f}{\partial y}\right)$

Back to Examples:

1) $f(x,y) = x^2y$

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2$$

2) $g(x,y) = \cos(x+y^3)$

$$\frac{\partial f}{\partial x} = -\sin(x+y^3)$$

$$\frac{\partial f}{\partial y} = -\sin(x+y^3)\cdot 3y^2$$

Partial Differential Equations

Equation(s) involving an unknown function f and its partial derivatives.

The Heat Equation

(one spatial variable)

$f = f(x,t)$, a function of position x and time t

The equation

$$\boxed{\frac{\partial f}{\partial t} = k \frac{\partial^2 f}{\partial x^2}}$$

is the heat equation, which governs the flow of heat through a homogeneous piece of material.

Here, k is the thermal conductivity of the material and $\frac{\partial^2 f}{\partial x^2}$ is the second partial of f:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

(the partial of the partial)

We want non-constant solutions.

How to solve your equation:

Assume $f(x,t) = g(x)h(t)$

for one-variable functions

g and h. Then

$$\frac{\partial f}{\partial t} = g(x)h'(t)$$

$$\frac{\partial^2 f}{\partial x^2} = g''(x)h(t).$$

Substituting, we have

$$g(x)h'(t) = k\,g''(x)h(t),$$

and rearranging,

$$\boxed{\frac{h'(t)}{k\,h(t)} = \frac{g(x)}{g''(x)}}.$$

Since x and t are independent variables, both sides of the equality must be equal to the same constant c.

We then have two ordinary differential equations,

i) $\dfrac{h'(t)}{k\,h(t)} = C$ and

ii) $\dfrac{g''(x)}{g(x)} = C$.

The first we know how to solve. What about the second?

Rewrite as

$$g''(x) = cg(x) \text{ and}$$

then as

$$g''(x) - cg(x) = 0.$$

Now let $\boxed{g(x) = e^{\lambda x}}$

for some constant λ.

Then $g'(x) = \lambda e^{\lambda x}$ and

$$g''(x) = \lambda^2 e^{\lambda x}.$$

Substituting, we get

$$\lambda^2 e^{\lambda x} - c e^{\lambda x} = 0$$

and factoring out $e^{\lambda x}$,

$$e^{\lambda x} (\lambda^2 - c) = 0 .$$

But $e^{\lambda x} > 0$, so we must have

$$\lambda^2 - c = 0 ,$$ that is,

$$\lambda = \pm \sqrt{c} .$$

In this way, the use of exponentials leads to solutions of the heat equation.